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Locally s-arc-transitive graphs related to sporadic simple groups

Dimitri Leemans

Université Libre de Bruxelles, Département de Mathématiques, Service de Géométrie – CP 216, Boulevard du Triomphe, B-1050 Bruxelles, Belgium

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Dedicated to John Cannon and Derek Holt, on the occasions of their significant birthdays, in recognition of distinguished contributions to mathematics

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A B S T R A C T

We show that, up to isomorphism, there are ten connected locally $(\mathrm{M}_{11}, 2)$-arc-transitive graphs. We compute, up to conjugacy, the connected locally (G, s)-arc-transitive graphs for $G = \mathrm{M}_{11}$, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, He, Ru, Suz, Co_3, and we obtain a partial classification for O′N. The classification provides locally $(G, 7)$- and $(G, 9)$-arc-transitive graphs for He and Ru, and locally $(G, 7)$-arc-transitive graphs for J_3 and O′N. We also obtain new cubic 5-arc-transitive graphs for He and Ru.

1. Introduction

Giudici, Li and Praeger initiated in [8] a program for studying locally $(G, 2)$-arc-transitive connected graphs where G is vertex-intransitive, using the same philosophy as Praeger's earlier work for 2-arc-transitive graphs [16]. All such graphs are bipartite with the two parts of the partition being G-orbits. It is shown in [8] that the basic graphs to study in this family are those where G is quasiprimitive on at least one of its two orbits and any other graph in the family is a cover of one of these basic graphs. The possible O'Nan–Scott types of the quasiprimitive actions were also determined and one possibility is where G is almost simple. Our paper contributes to this program by classifying all locally $(G, 2)$-arc-transitive graphs where G is one of 14 sporadic simple groups, namely the five Mathieu groups, the first three Janko groups, HS, McL, He, Ru, Suz and Co_3. Moreover, it gives an almost complete classification for O′N. We also obtain the vertex-transitive $(G, 2)$-arc transitive graphs for these groups via the G-vertex-intransitive locally 2-arc transitive graphs that we classify.

E-mail address: dleemans@ulb.ac.be.

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As we shall see in Section 3, it is possible to classify these graphs by hand for the smallest sporadic group, namely M_{11}, once we know the full subgroup lattice of this group. The lattice is available, for instance in [3]. We get a first result.

Theorem 1.1. *Up to conjugacy (and therefore up to isomorphism), there are ten locally* $(\mathsf{M}_{11}, 2)$*-arc-transitive graphs.*

Its proof shows that such a classification is lengthy to obtain by hand and suggests using a computer to obtain similar results for bigger sporadic groups. In Section 4, we describe briefly Magma [1] programs to classify all locally $(G, 2)$-arc-transitive graphs for a given permutation group G. We give the results obtained for the 15 smallest sporadic groups. We obtain a complete classification for each of these groups except for O′N.

Once we know the locally $(G, 2)$-arc-transitive graphs of G, we may easily determine with Magma the highest value of s for which they are locally (G, s)-arc-transitive. This is done in Section 5 where we also give a necessary condition for a graph to be locally (G, s)-arc-transitive.

We give more details on all the locally (G, s)-arc-transitive graphs obtained with $s \geqslant 4$. Some spectacular examples arise. For instance, we get new locally $(G, 7)$- and $(G, 9)$-arc-transitive graphs for $G = \mathsf{He}$ and Ru. We also get some locally $(G, 7)$-arc-transitive graphs for J_3 and O′N. Moreover, the locally $(G, 9)$-arc-transitive graphs for Ru and He also give new examples of cubic 5-arc-transitive graphs.

A necessary step for the classification is to obtain the subgroup lattice of a group. Critical to the success of this computation is the algorithm of Cannon and Holt [6] to obtain the maximal subgroups of a group. Over the past seven years, we have developed and refine programs to exploit this work and to improve the efficiency of the lattice computation. We are now able to compute (full or partial) lattices for Suz, Ru, O′Nan, Co_3 and Fi_{22}. Recently, Cannon (personal communication) pointed out more improvements due to Bill Unger which dramatically speed up computing the full subgroup lattice of a permutation group.

The results obtained in this paper open several paths for future research. Due to the large number of locally $(G, 2)$- and $(G, 3)$-arc-transitive graphs obtained, we feel it is too ambitious to try to obtain such a classification for all of the 26 sporadic simple groups. However, classifying the locally (G, s)-arc-transitive graphs with $s \geqslant 4$ seems more reachable. Such a classification already exists for finite vertex-primitive and vertex-biprimitive locally s-arc-transitive graphs (see [15]).

2. Definitions and notation

2.1. Locally s-arc-transitive graphs

The following definitions are taken from [8].

Let $\mathcal{G}$ be a finite simple undirected connected graph. Denote by V (resp. E) its vertex-set (resp. edge-set). The edge-set may be identified with a subset of unordered pairs from V. An s-arc is an ordered $(n+1)$-tuple $(\alpha_0, \ldots, \alpha_n)$ of vertices such that $\{\alpha_{i-1}, \alpha_i\}$ is an edge of $\mathcal{G}$ for all $i = 1, \ldots, n$ and $\alpha_{j-1} \neq \alpha_{j+1}$ for all $j = 1, \ldots, n-1$. Let G be a subgroup of the automorphism group $Aut(\mathcal{G})$ of $\mathcal{G}$. The graph $\mathcal{G}$ is said to be (G, s)*-arc-transitive* if G is transitive on the set of s-arcs of $\mathcal{G}$; also, $\mathcal{G}$ is said to be s*-arc-transitive* if it is $(Aut(\mathcal{G}), s)$-arc-transitive. Similarly $\mathcal{G}$ is said to be $(G, 1)$*-arc-transitive* if G is transitive on the 1-arcs of $\mathcal{G}$, that is on the ordered pairs (α_0, α_1) where $\{\alpha_0, \alpha_1\}$ is an edge of $\mathcal{G}$. Obviously, 1-arc transitivity is equivalent to flag-transitivity if the graph $\mathcal{G}$ is seen as a rank two geometry whose elements of type 0 (resp. 1) are the vertices (resp. edges) of $\mathcal{G}$. Moreover, 2-arc transitivity is equivalent to property $(2T)_1$ described in [5].

If all vertices have valency at least two, local (G, s)-arc-transitivity implies local $(G, s-1)$-arc-transitivity. Therefore, if we want to classify all connected locally (G, s)-arc-transitive graphs with $s \geqslant 2$ for a given group G, a good starting point is to generate all the connected locally $(G, 2)$-arc-transitive graphs for G.

The search for all connected $(G, 2)$-arc-transitive graphs is equivalent to determining the pairs of subgroups $\{G_0, G_1\}$ in G such that

- $B := G_0 \cap G_1$ is a subgroup of index 2 in G_1 (this ensures that the coset geometry $\Gamma(G, \{G_0, G_1\})$ is a graph);
- G_0 has a 2-transitive action on the cosets of B in G_0 (this ensures 2-arc transitivity);
- $\langle G_0, G_1 \rangle = G$ (this ensures connectedness of the graph);
- G_0 is core-free in G.

Given $G \leqslant Aut(\mathcal{G})$, we call $\mathcal{G}$ *locally* (G, s)*-arc-transitive* if $\mathcal{G}$ contains an s-arc and given any two s-arcs α and β starting at the same vertex v, there exists an element $g \in G_v$ mapping α to β. We say $\mathcal{G}$ is *locally s-arc-transitive* if it is locally (G, s)-arc-transitive for some $G \leqslant Aut(\mathcal{G})$.

Following [8], the search for graphs having G acting as a locally 2-arc-transitive automorphism group is equivalent to determining the pairs of subgroups $\{G_0, G_1\}$ in G such that

(P_1) G_0 (resp. G_1) has a 2-transitive action on the cosets of B in G_0 (resp. G_1) (this ensures local 2-arc transitivity);
(P_2) $\langle G_0, G_1 \rangle = G$ (this ensures connectedness of the graph);
(P_3) $G_0 \cap G_1$ is core-free in G.

For each pair of subgroups $\{G_0, G_1\}$ satisfying (P_1), (P_2) and (P_3), the corresponding graph is obtained by taking as vertex-set the set of right cosets of G_0 and G_1 in G. Two cosets are adjacent provided their intersection is non-empty.

If $\{[G_0 : B], [G_1 : B]\} = \{2, k\}$, we obtain a bipartite graph Γ with one part of valency 2 and the other of valency k. As shown in [8], if Γ is locally $(G, 2s - 1)$-arc-transitive, a (G, s)-arc-transitive graph can be constructed from Γ by taking the distance-two graph induced on the vertices not of valency two, provided that Γ is not isomorphic to K_{2k}. Conversely, for $s \geqslant 2$, if Σ is a (G, s)-arc-transitive graph, by placing one vertex at the midpoint of each edge of Σ, we obtain a locally $(2s - 1)$-arc-transitive graph. These graphs are in 1–1 correspondence as shown by Corollary 3.11 of [8]. Therefore, if we classify all the pairs of subgroups of a given group G satisfying properties (P_1), (P_2) and (P_3), we can construct all the 2-arc-transitive graphs for the same group G as well.

We call a locally $(G, 2)$-arc-transitive graph *thick* if all of its vertices have valency at least three.

2.2. Coset geometries and diagrams

The aim of this section is to explain the diagrams of Section 3. They give some more information on the locally $(\mathsf{M}_{11}, 2)$-arc-transitive graphs. The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [19] (see also [4, Chapter 3]).

Let $\Gamma(X, *, t, I)$ be an incidence structure. A *flag* of Γ is a set of elements of X that are pairwise incident. A *chamber* is a flag containing one element of each type. The *rank* of Γ is $|I|$.

In this paper, we construct locally $(G, 2)$-arc-transitive graphs, which consist of a group G and a pair of subgroups $\{G_0, G_1\}$ of G satisfying properties (P_1), (P_2) and (P_3). The following theorem, due to Jacques Tits, shows that these locally $(G, 2)$-arc-transitive graphs are rank two incidence structures.

Theorem 2.1. *(See Tits, 1962 [19].) Let n be a positive integer. Let $I := \{0, \ldots, n - 1\}$ be a finite set and let G be a group together with a family of subgroups $(G_i)_{i \in I}$. Let X be the set consisting of all cosets $G_i g$, $g \in G$, $i \in I$. Let $t : X \to I$ be defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by:*

$$G_i g_1 * G_j g_2 \quad \text{if and only if} \quad G_i g_1 \cap G_j g_2 \text{ is non-empty in } G.$$

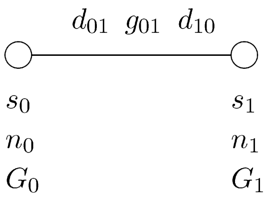
The 4-tuple $\Gamma := (X, *, t, I)$ is an incidence structure having a chamber. Moreover, G acts by right multiplication as an automorphism group on Γ. Finally, G is transitive on the flags of rank less than 3.

We usually write $\Gamma := \Gamma(G; (G_i)_{i \in I})$ to denote the incidence structure constructed from the group G and the subgroups G_i using Theorem 2.1.

The *incidence graph* of Γ is the graph whose vertices are the right cosets of the subgroups $(G_i)_{i\in I}$. Two vertices are joined provided the corresponding cosets have a non-empty intersection. The *type* of a vertex $v = G_i g$ of the incidence graph is i. Here, we assume that $I := \{0, 1\}$. Observe that the incidence graph is exactly the locally 2-arc-transitive graph we constructed in the previous section.

The *Buekenhout diagram* of Γ is a graph on the set $I := \{0, 1\}$ together with the following structure: to each vertex $i \in I$, we attach the *order* $s_i := v_j - 1$ where v_j is the valency of a vertex of type $j \neq i$, the *number* n_i *of vertices of type* i, which is the index of G_i in G, and the subgroup G_i. On the edge joining 0 to 1, there are three numbers, d_{01}, g_{01}, d_{10}, where g_{01} (the *gonality*) is equal to half the girth of the incidence graph, and d_{01} (resp. d_{10}), the 0-*diameter* (resp. 1-*diameter*) is the greatest distance from some fixed 0-element (resp. 1-element) to any other element in the incidence graph.

On a picture of the diagram, this structure is depicted as follows.



If $g_{01} = d_{01} = d_{10} - 1 = 3$ and $s_0 = 1$, we write c in place of these three numbers because the residue is isomorphic to a complete graph.

3. M_{11} and its locally 2-arc-transitive graphs

There are already four locally $(\mathsf{M}_{11}, 2)$-arc-transitive graphs known for the Mathieu group M_{11}. These are the rank two geometries of M_{11} given in [14]. They were discovered by Michel Dehon and Xavier Miller in 1996. In this section, we prove Theorem 1.1. We give in Fig. 1 the Buekenhout diagrams [2] of the 10 locally $(\mathsf{M}_{11}, 2)$-arc-transitive graphs. Graphs Γ_2, Γ_4, Γ_6 and Γ_7 are respectively geometries 2.1 to 2.4 in [14]. These diagrams have been computed using the Coset Geometry package of MAGMA. One could compute them by hand but we feel it is not relevant to the paper to give all the details here, so we leave this as an exercise to the interested reader. Observe that, since $Aut(\mathsf{M}_{11}) = \mathsf{M}_{11}$, the number of graphs obtained up to conjugacy is equal to the number of graphs up to isomorphism.

Proof of Theorem 1.1. First, we determine the possible groups G_0. We may assume without loss of generality that these are subgroups of M_{11} having a subgroup B of index at least three, and such that G_0 acts 2-transitively on the cosets of B in G_0. We use the classification of the 2-transitive permutation groups (see [9] for instance). It gives us the possibilities for B once G_0 is fixed. We search for possible subgroups G_1 and without loss of generality, we may assume that the order of G_1 is at most the order of G_0. We proceed in decreasing order for G_0. Looking at the subgroup lattice of M_{11} given in [3], we get the following cases.

1. $G_0 \cong \mathsf{M}_{10}$ implies that $B \cong \mathsf{M}_9$ and $G_1 \cong \mathsf{M}_{10}$ or $\mathsf{M}_9 : 2$. This gives us graphs Γ_1 and Γ_2.
2. $G_0 \cong L_2(11)$ implies that $B \cong A_5$ or $11 : 5$. The latter is in a unique proper subgroup of M_{11} and hence cannot give a locally 2-arc-transitive graph. There are two classes of A_5-subgroups in M_{11}, one of length 66 and the other of length 132. The first gives two possibilities for G_1, namely $L_2(11)$ and S_5. The corresponding graphs are Γ_3 and Γ_4. The orbit of length 132 of A_5-subgroups gives another possibility for G_1, namely A_6. The corresponding graph is Γ_5.
3. $G_0 \cong A_6$ implies that B is isomorphic to either A_5 or $3^2 : 4$. The first case gives no possible G_1. If $B \cong 3^2 : 4$, the only possible subgroup for G_1 is isomorphic to M_9. Therefore $\langle G_0, G_1\rangle = \mathsf{M}_{10}$.
4. $G_0 \cong \mathsf{M}_9 : 2$ implies that $B \cong \Gamma L_1(9)$ and $G_1 \cong GL_2(3)$. This is graph Γ_6.
5. $G_0 \cong S_5$ gives $B \cong S_4$ or $5 : 4$. Both cases give no possibility for G_1.

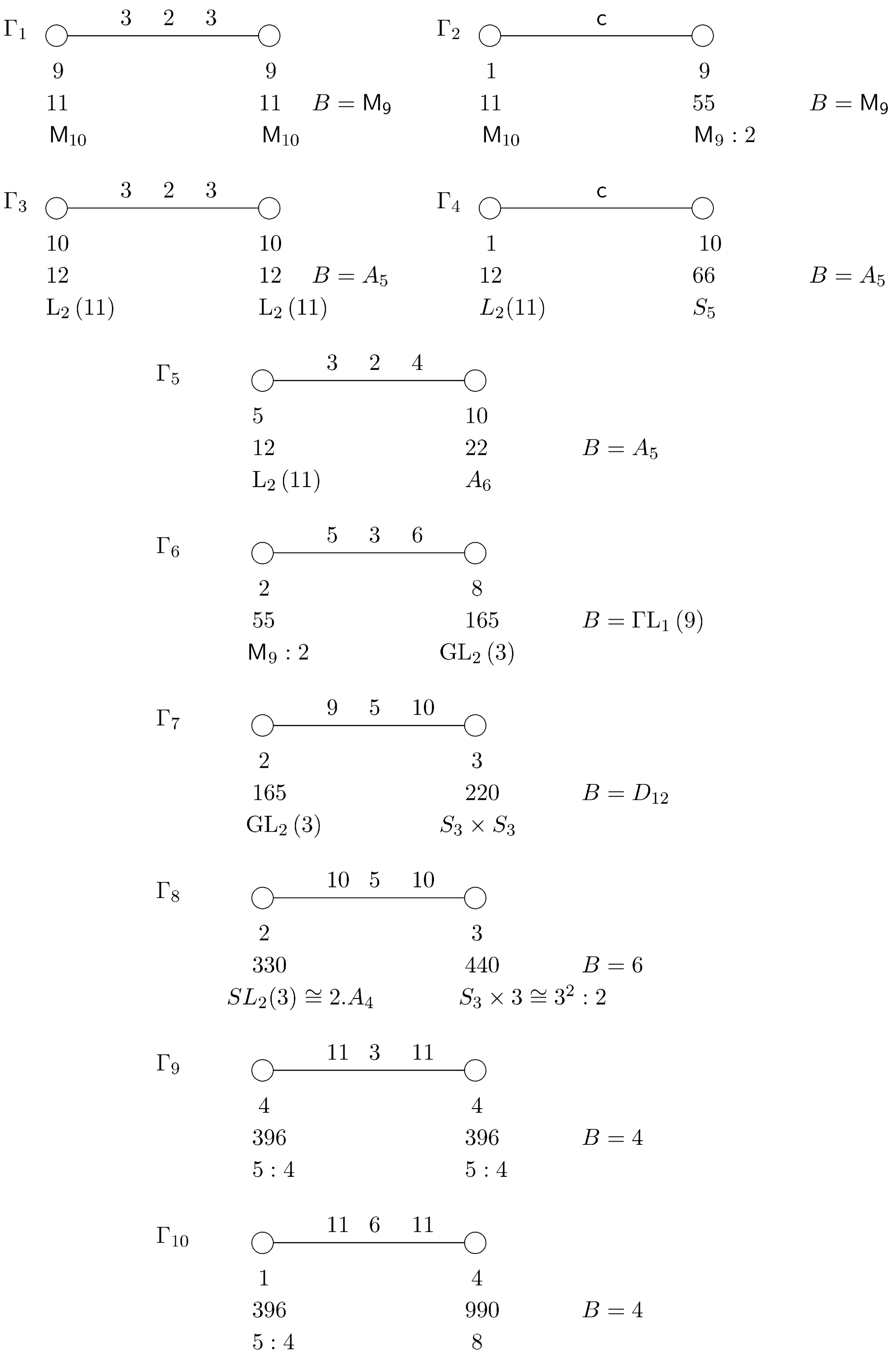


Fig. 1. The ten locally $(\mathsf{M}_{11}, 2)$-arc-transitive graphs.

6. $G_0 \cong 3^2:8$ gives $B \cong Z_8$ and $G_1 \cong \Gamma L_1(9)$. Therefore $\langle G_0, G_1\rangle = \mathsf{M}_9:2$.
7. $G_0 \cong \mathsf{M}_9$ gives $B \cong Q_8$ and $G_1 \cong \mathsf{M}_9$ or $G_1 \cong \Gamma L_1(9)$. But $\langle G_0, G_1\rangle \cong \mathsf{M}_{10}$ or $\mathsf{M}_9:2$ and the corresponding graph is not connected.
8. $G_0 \cong 3^2:D_8$ has only maximal subgroups of index 2, so there is no possibility for B in this case.
9. $G_0 \cong A_5$. There are two classes of A_5. The possible B-subgroups are isomorphic to A_4 or D_{10}. One may easily check using the ternary Golay code associated to M_{11} that no two A_5-subgroups meeting in an A_4 or a D_{10} may generate M_{11}. Other possibilities with $B \cong A_4$ and $G_1 \cong S_4$ or $B \cong D_{10}$ and $G_1 \cong 5:4$ both give $\langle G_0, G_1\rangle < \mathsf{M}_{11}$.
10. $G_0 \cong 11:5$ has obviously no 2-transitive permutation representation.
11. $G_0 \cong GL_2(3)$ gives $B \cong D_{12}$ or $\Gamma L_1(9)$. The latter case gives no choice for G_1. The former case gives $G_1 \cong S_3 \times S_3$. This is the graph Γ_7.
12. $G_0 \cong 3^2:4$ gives no possibility for B.
13. $G_0 \cong S_3 \times S_3$ implies $B \cong D_{12}$ and no possibility for G_1.
14. $G_0 \cong S_4$ implies $B \cong D_8$ or D_6.
    The case where $B \cong D_8$ yields $G_1 \cong S_4$ or $\Gamma L_1(9)$. One may easily show that two subgroups S_4 meeting in a D_8 must have a fixed point in common in the usual representation on 11 points of M_{11} and hence they may generate at most a subgroup M_{10}. The same is true if $G_1 \cong \Gamma L_1(9)$.
    The case where $B \cong D_6$ gives $G_1 \cong S_4$, $3^2:2$ or D_{12}. If $G_1 \cong S_4$, one may easily show that $\langle G_0, G_1\rangle \cong S_5$. If $G_1 \cong 3^2:2$, the subgroup lattice tells us that $\langle G_0, G_1\rangle = A_6$. Finally, if $G_1 \cong D_{12}$, we have $\langle G_0, G_1\rangle \cong S_5$.
15. $G_0 \cong SL_2(3) = 2\cdot A_4$ implies that $B \cong Z_6$. Therefore $G_1 \cong S_3 \times 3$ gives graph Γ_8.
16. $G_0 \cong 5:4$ gives $B \cong Z_4$ and $G_1 \cong 5:4$, Z_8, Q_8 or D_8.
    The first case gives graph Γ_9. One may check that the $5:4$-subgroups correspond to stabilizers of a block of the $S(4,5,11)$ and a point not in the block. Incidence is defined by saying that an element x_0 of type 0 is incident to an element x_1 of type 1 if the block of x_0 has one point in common with the block of x_1 and the points of x_0 and x_1 are distinct and outside both blocks. This implies that $\langle G_0, G_1\rangle = \mathsf{M}_{11}$.
    The last two cases give $\langle G_0, G_1\rangle \cong \mathsf{M}_{10}$ or S_5.
    The case where $G_0 \cong Z_8$ gives graph Γ_{10}.
17. $G_0 \cong S_3 \times 3$ implies $B \cong Z_6$. Hence $G_1 \cong D_{12}$ and $\langle G_0, G_1\rangle \cong S_3 \times S_3$.
18. $G_0 \cong 3^2:2$ implies $B \cong S_3$ and the same argument as in case 17 applies.
19. The remaining candidates for G_0 are too small to be able to find a G_1 such that $\langle G_0, G_1\rangle = \mathsf{M}_{11}$. $\square$

4. The search for locally $(G,2)$-arc-transitive graphs

To classify locally 2-arc-transitive graphs for a given group G requires knowledge of the conjugacy classes of its subgroups. The classification can be most accurately performed by machine. We have developed software in MAGMA for this task. Two pairs of subgroups $\{G_0, G_1\}$ and $\{G_0', G_1'\}$ are *conjugate* provided there exists an element $g \in G$ such that either $G_0^g = G_0'$ and $G_1^g = G_1'$ or $G_1^g = G_0'$ and $G_0^g = G_1'$. We classify the graphs up to conjugacy under the action of G on pairs of subgroups satisfying properties (P_1), (P_2) and (P_3). We give one representative for each conjugacy class.

Fig. 2 gives a rough description of the algorithm. In fact, our implementation constructs a permutation group acting on the minimal overgroups of B in order to obtain one representative of each orbit of pairs of subgroups $\{G_0, G_1\}$ faster.

In Table 1, we give for each of the 15 smallest sporadic simple groups its name (G), its order (Order(G)), its smallest permutation representation degree (Deg(G)), the number of conjugacy classes of subgroups it has ($cc(G)$), the number of locally 2-arc-transitive graphs it has up to conjugacy (# l-2-arc-trans), the number of thick locally 2-arc-transitive graphs it has up to conjugacy (# thick) and the number of locally 2-arc-transitive graphs with one valency equal to 2 (# 2-arc-trans). The times taken on a workstation with two Intel Xeon dual core 64 bits processors at 3.2 GHz and 16 Gigabytes of memory range from 0.7 seconds (for M_{11}) to 24.8 days (for O′N). For O′N, we could not complete the computation for the class of subgroups of order 2. In [11], we already found three (O′N, 2)-arc-transitive and seven locally (O′N, 2)-arc-transitive graphs.

```
Input : G, a permutation group
Output : S2, a set containing pairs of subgroups {G0, G1} of G satisfying properties P1 to P3

Compute the subgroup lattice of G
Let S2 be an empty set in which we will store the pairs of subgroups corresponding
  to locally 2-arc-transitive graphs.
For each representative B <> Id(G) of a conjugacy class of subgroups of G with B corefree do
  Let S be an empty set.
  Compute a set L containing all the minimal overgroups M of B that act
    2-transitively on the cosets of B in M
  For each subgroup M in L, add M to S provided no subgroup in S is conjugated to M in G
    and B has index at least three in M.
  For each subgroup M in S do
    For each subgroup N in L do
      if <M, L> = G and {M, N} is not conjugate to any element of S2 in G
        add {M,L} to S2

return S2;
```

Fig. 2. An algorithm to compute locally 2-arc-transitive graphs of a group G.

Table 1
The number of locally $(G,2)$-arc-transitive graphs for some sporadic groups G up to conjugacy.

G	Order(G)	Deg(G)	$cc(G)$	# l-2-arc-trans	# thick	# 2-arc-trans
M_{11}	7,920	11	39	10	7	3
M_{12}	95,040	12	147	45	39	6
J_1	175,560	266	40	43	28	15
M_{22}	443,520	22	156	27	22	5
J_2	604,800	100	146	69	58	11
M_{23}	10,200,960	23	204	16	11	5
HS	44,352,000	100	589	38	34	4
J_3	50,232,960	6156	137	185	155	30
M_{24}	244,823,040	24	1529	141	134	7
McL	898,128,000	275	373	31	29	2
He	4,030,387,200	2058	1698	464	372	92
Ru	145,926,144,000	4060	6035	769	627	142
Suz	448,345,497,600	1782	6381	1395	1233	162
O′N	460,815,505,920	122760	581	685+	537+	148+
Co_3	495,766,656,000	276	2483	324	296	28

5. Locally (G,s)-arc-transitive graphs

Once we know all the locally $(G,2)$-arc-transitive graphs, we may easily check with Magma which ones are locally (G,s)-arc-transitive for $s = 3, 4, \ldots$. In Tables 2 and 3, for each of the 15 sporadic groups, we state how many of the graphs are locally (G,s)-arc-transitive but not locally $(G,s+1)$-arc-transitive.

Table 2 deals with the thick locally $(G,2)$-arc-transitive graphs. Table 3 deals with the non-thick locally $(G,2)$-arc-transitive graphs, i.e. the graphs with one valency equal to 2. For O′N, we write a '+' sign after the numbers of graphs to remind the reader that the classification is not complete in that case. Although the case $B = 2$ is not done, we have the exact numbers in the thick case for $s > 2$ and in the non-thick case for $s > 3$. This is implied by the following lemma and its corollary, direct consequences of Lagrange's theorem.

Lemma 5.1. *Let G be a group and $\{G_0, G_1\}$ be a pair of subgroups of G satisfying properties (P_1), (P_2) and (P_3). Denote by v_i the index of $B := G_0 \cap G_1$ in G_i (with $i = 0, 1$). If $(G; \{G_0, G_1\})$ is a locally s-arc-transitive graph (with $s \geqslant 2$), then*

Table 2
Number of thick locally s-arc-transitive graphs which are not locally $(s+1)$-arc-transitive.

G	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
M_{11}	4	3				
M_{12}	17	20	1	1		
M_{22}	22					
M_{23}	7	4				
M_{24}	83	47	1	3		
J_1	24	4				
J_2	34	22	0	2		
J_3	129	21	3	2		
HS	16	18				
McL	22	7				
He	275	85	4	6	0	2
Ru	537	85	4	1		
Suz	1030	199	4			
O′N	417+	116	4			
Co_3	244	51	0	1		

Table 3
Number of non-thick locally s-arc-transitive graphs which are not locally $(s+1)$-arc-transitive.

G	$s=3$	$s=5$	$s=7$	$s=9$
M_{11}	3			
M_{12}	5	1		
M_{22}	5			
M_{23}	5			
M_{24}	7			
J_1	15			
J_2	11			
J_3	26	1	3	
HS	4			
McL	2			
He	78	8	4	2
Ru	132	4	5	1
Suz	157	5		
O′N	141+	5	2	
Co_3	28			

- $((v_0-1)(v_1-1))^{\frac{s-1}{2}}$ *divides* $|B|$ *if* s *is odd*;
- $((v_0-1)(v_1-1))^{\frac{s-2}{2}} \cdot lcm(v_0-1, v_1-1)$ *divides* $|B|$ *if* s *is even,*

where $lcm(v_0-1, v_1-1)$ *is the lowest common multiple of* v_0-1 *and* v_1-1.

Corollary 5.1. *If* $(G; \{G_0, G_1\})$ *is a locally* s*-arc-transitive graph with* $B := G_0 \cap G_1$ *a cyclic group of prime order, then* s *is at most* 3*. Moreover, if* $s = 3$*, then one of* v_0 *or* v_1 *must be equal to* 2.

This corollary shows that, for O′N, the classification is complete in the thick case for $s > 2$ and in the non-thick case for $s > 3$.

Observe that the necessary condition given by Lemma 5.1 is not sufficient. For instance, take $G := Alt(n)$ acting on its standard permutation representation on $\Omega := \{1, \dots, n\}$. Take as vertex stabilizers the stabilizer of 1 in G and the stabilizer of $\{1, 2\}$ in G. They are respectively isomorphic to $Alt(n-1)$ and $Sym(n-2)$. Their intersection is isomorphic to $Alt(n-2)$. The graph $\mathcal{G}$ obtained is a locally $(Alt(n), 3)$-arc-transitive graph, it is not locally $(Alt(n), 4)$-arc-transitive, and the condition of Lemma 5.1 is satisfied for values of s greater than 3 whenever $n-2$ is not a prime and $n > 6$. Indeed, for $n > 6$, if $n-2$ is not a prime, then $(n-2)^2 \mid (n-2)!$.

MAGMA files containing the permutation generators of the pairs of subgroups corresponding to all the graphs found in this paper (except for O′N, due to disk space) are available at the following address:

http://cso.ulb.ac.be/~dleemans/abstracts/lsarc.html

When loading one of these files in MAGMA for a group G, we get a sequence geo of pairs of subgroups corresponding to the locally $(G, 2)$-arc-transitive graphs.

In Table 4, we give for a group G, an index n in the corresponding sequence geo, the number s such that the graph $\mathcal{G}_n$ is locally (G, s)-arc-transitive but not locally $(G, s+1)$-arc-transitive, the sta-

Table 4
Locally (G, s)-arc-transitive graphs with $s \geqslant 4$.

G	s	G_0	G_1	B	v_0	v_1	A_0	A_1	n	Ref.
M_{12}	4	$AGL_2(3)$	$AGL_2(3)$	$3^2 : D_{12}$	4	4	2^2	2^2	37	[17]
	5	$M_8 : S_4$	$2^{2+3} : S_3$	$Q_8 : D_8$	3	3	2^2	2^2	39	[17]
		$3^2 : 2$	$A_4 \times 3$	3^2	2	4	Id	Id	16	
M_{24}	4	S_4	S_4	D_8	3	3	Id	Id	94	
	5	$S_4 \times 2$	$S_4 \times 2$	$D_8 \times 2$	3	3	Id	Id	109	
		$A_5 \times A_4$	$2^4 : (A_4 \times 3)$	$A_4 \times A_4$	5	4	Id	Id	131	
		$A_5 \times A_4 : 2$	$2^4 : (A_4 \times 3) : 2$	$(A_4 \times A_4) : 2$	5	4	2	2	134	
J_2	5	$A_5 \times A_4$	$2^{2+4} : 3^2$	$A_4 \times A_4$	5	4	Id	Id	67	[12]
		$2^{1+4} \cdot A_5$	$2^{2+4} \cdot 3 \times S_3$	$2^{1+4} \cdot A_4$	5	3	6	6	69	[7]
J_3	4	S_4	S_4	D_8	3	3	Id	Id	151	
		$2^4 : A_5$	$2^4 : A_5$	$2^{2+4} : 3$	5	5	3	3	181	
		$2^4 : 3 \times A_5$	$2^4 : 3 \times A_5$	$2^4 : 3 \times A_4$	5	5	3^2	3^2	185	[13]
	5	$(Q_8 : 3) : 4$	$2^{2+2} : 6$	$4^2 : 2$	3	3	2	2	177	
		$2^{1+4} : A_5$	$2^{2+4} : 3 \times S_3$	$2^{1+4} : A_4$	5	3	6	6	183	[13]
		$3^2 : 2$	$3 \times A_4$	3^2	2	4	Id	Id	137	
	7	$D_8 \cdot 2$	S_4	D_8	2	3	Id	Id	148	
		$2^4 : A_5$	$2^{2+4} : 6$	$2^{2+4} : 3$	5	2	3	3	180	
		$2^4 : 3 \times A_5$	$2^{2+4} : 3 \times S_3$	$2^4 : 3 \times S_3$	2	5	3^2	3^2	184	[20]
He	4	S_4	S_4	D_8	3	3	Id	Id	341	
		S_4	S_4	D_8	3	3	Id	Id	344	
		S_4	S_4	D_8	3	3	Id	Id	345	
		S_4	S_4	D_8	3	3	Id	Id	347	
	5	$S_4 \times 2$	$S_4 \times 2$	$D_8 \times 2$	3	3	Id	Id	383	
		$S_4 \times 2$	$S_4 \times 2$	$D_8 \times 2$	3	3	Id	Id	385	
		$S_4 \times 2$	$S_4 \times 2$	$D_8 \times 2$	3	3	Id	Id	386	
		$S_4 \times 2$	$S_4 \times 2$	$D_8 \times 2$	3	3	Id	Id	389	
		$2^3 \cdot S_4$	$2^3 \cdot S_4$	$2^3 \cdot D_8$	3	3	2^2	2^2	437	
		$2^3 \cdot S_4$	$2^3 \cdot S_4$	$2^3 \cdot D_8$	3	3	2^2	2^2	438	
		D_{12}	D_8	2^2	3	2	Id	Id	216	
		D_{12}	D_8	2^2	3	2	Id	Id	217	
		D_{12}	D_8	2^2	3	2	Id	Id	218	
		D_{12}	D_8	2^2	3	2	Id	Id	219	
		D_{12}	D_8	2^2	3	2	Id	Id	240	
		D_{12}	D_8	2^2	3	2	Id	Id	241	
		D_{12}	D_8	2^2	3	2	Id	Id	242	
		D_{12}	D_8	2^2	3	2	Id	Id	243	
	7	$2^4 \cdot S_4$	$2^3 \cdot (2 \times S_4)$	$2^4 \cdot D_8$	3	3	2	2	443	
		$2^4 \cdot S_4$	$2^3 \cdot (2 \times S_4)$	$2^4 \cdot D_8$	3	3	2	2	443	
		$D_8 \cdot 2$	S_4	D_8	2	3	Id	Id	339	
		$D_8 \cdot 2$	S_4	D_8	2	3	Id	Id	342	
		D_{16}	S_4	D_8	2	3	Id	Id	340	
		D_{16}	S_4	D_8	2	3	Id	Id	343	
	9	$(2 \times D_8) : 2$	$S_4 \times 2$	$2 \times D_8$	2	3	Id	Id	382	
		$(2 \times D_8) : 2$	$S_4 \times 2$	$2 \times D_8$	2	3	Id	Id	384	

Table 4 (*continued*)

G	s	G_0	G_1	B	v_0	v_1	A_0	A_1	n	Ref.
Ru	4	$5^2:4\cdot S_5$	$5^2:4\cdot S_5$	$5^2:4\cdot(5:4)$	6	6	4^2	4^2	763	
		S_4	S_4	D_8	3	3	Id	Id	659	
		S_4	S_4	D_8	3	3	Id	Id	658	
		S_4	S_4	D_8	3	3	Id	Id	639	
	5	$2\times S_4$	$2\times S_4$	$2\times D_8$	3	3	Id	Id	700	
		D_{12}	D_8	2^2	3	2	Id	Id	443	
		D_{12}	D_8	2^2	3	2	Id	Id	444	
		$5:4^2$	$4^2:2$	4^2	5	2	Id	Id	697	
		A_7	$PGL(2,9)$	A_6	7	2	D_{10}	D_{10}	753	
	7	S_4	D_{16}	D_8	3	2	Id	Id	638	
		S_4	D_{16}	D_8	3	2	Id	Id	654	
		S_4	D_{16}	D_8	3	2	Id	Id	655	
		S_4	$8:2$	D_8	3	2	Id	Id	653	
		$5^2:4\cdot S_5$	$N_{\mathrm{Ru}}(5^2:4\cdot 5:4)$	$5^2:4\cdot 5:4$	6	2	4^2	4^2	762	[18]
	9	$2\times S_4$	$(2\times D_8):2$	$2\times D_8$	3	2	Id	Id	699	
Suz	4	S_4	S_4	D_8	3	3	Id	Id	599	
		S_4	S_4	D_8	3	3	Id	Id	600	
		S_4	S_4	D_8	3	3	Id	Id	601	
		S_4	S_4	D_8	3	3	Id	Id	602	
	5	D_{12}	D_8	2^2	3	2	Id	Id	297	
		D_{12}	D_8	2^2	3	2	Id	Id	298	
		D_{12}	D_8	2^2	3	2	Id	Id	306	
		D_{12}	D_8	2^2	3	2	Id	Id	307	
		$(6\times 2):S_3$	$3^2:4$	$3^2:2$	3	2	2	2	623	
Co_3	5	$2^4:(A_4\times 3)$	$A_5\times A_4$	$A_4\times A_4$	4	5	Id	Id	320	

Table 5
Locally $(\mathrm{O'N}, s)$-arc-transitive graphs with $s \geqslant 4$.

s	G_0	G_1	B	v_0	v_1	A_0	A_1
4	$S_4=C_{471}$	$S_4=C_{471}$	$D_8=C_{556}$	3	3	Id	Id
4	$S_4=C_{471}$	$S_4=C_{473}$	$D_8=C_{556}$	3	3	Id	Id
4	$S_4=C_{471}$	$S_4=C_{473}$	$D_8=C_{556}$	3	3	Id	Id
4	$S_4=C_{473}$	$S_4=C_{473}$	$D_8=C_{556}$	3	3	Id	Id
5	$(3^2:4\times A_6)\cdot 2=C_{11}$	$3^4:2^{1+4}\cdot 2=C_{26}$	$(3^2:4\times 3^2:4)\cdot 2=C_{36}$	5	2	$C_{442}=2^2\cdot 2^3$	C_{442}
5	$(3^2:2\times A_6)\cdot 2=C_{19}$	$3^4:2^{1+3}\cdot 2=C_{38}$	$3^4:4^2=C_{71}$	5	2	$C_{518}=4\times 2^2$	C_{518}
5	$(3^2:2\times A_6)\cdot 2=C_{19}$	$3^4:2^{1+3}\cdot 2=C_{40}$	$3^4:4^2=C_{71}$	5	2	$C_{518}=4\times 2^2$	C_{518}
5	$D_8=C_{556}$	$D_{12}=C_{539}$	$2^2=C_{577}$	2	3	Id	Id
5	$D_8=C_{556}$	$D_{12}=C_{540}$	$2^2=C_{577}$	2	3	Id	Id
7	$S_4=C_{471}$	$D_8:2=C_{511}$	$D_8=C_{556}$	3	2	Id	Id
7	$S_4=C_{473}$	$D_8:2=C_{513}$	$D_8=C_{556}$	3	2	Id	Id

bilizer G_0 (resp. G_1) of a vertex x_0 (resp. x_1) such that x_0 is adjacent to x_1, the stabilizer $B := G_0 \cap G_1$ of the arc $\{x_0, x_1\}$, the valency v_0 (resp. v_1) of vertex x_0 (resp. x_1), the s-arc-stabilizer A_0 (resp. A_1) at vertex x_0 (resp. x_1), and a reference when we found one.

6. The O'Nan group

We now summarize our results for O'N. We have not completed the classification of the locally $(\mathrm{O'N}, 2)$-arc-transitive graphs: the principal difficulty is the large number of minimal overgroups. As previously observed, we have a complete classification of the locally $(\mathrm{O'N}, s)$-arc-transitive graphs for $s \geqslant 4$; these are summarized in Table 5, where we record structure and conjugacy class information from [10]. For $s = 3$, all missing graphs are not thick.

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